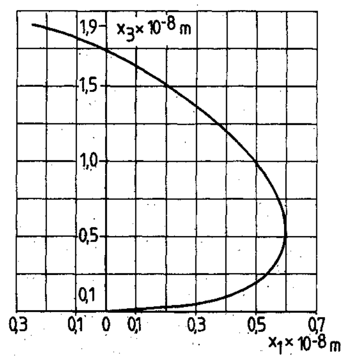

Fig. 1 Optimal transfer trajectory.

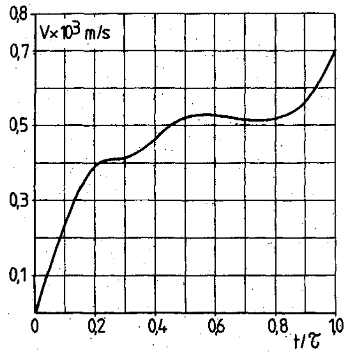

Fig. 2 Variation of the velocity.

where $x_1(t)$ and $x_3(t)$ represent the parametric form of the trajectory for the considered problem.

Numerical Application

On the basis of the analytical solution [Eq. (27)] a numerical application was performed concerning the optimal transfer from the libration point L_2 of the Earth-moon system.

It was taken into account that, for the Earth-moon system, $m^* = 0.01227$. The initial data used are $x_j(0) = 0$ ($j = 1, \ldots, 4$).

We mention that the Eqs. (27) were calculated in units $D = 1$ where $D = 3.84 \times 10^8$ m represents the Earth-moon distance and the time unit τ is chosen such that the gravitational parameter of the Earth is 1. The obtained results have been subsequently transformed in units m/s (Figs. 1 and 2). The evolution time was taken $T = \tau = 3.77 \times 10^5$ s and the components of the final velocity $c_2 = 400$ m/s and $c_4 = 600$ m/s.

Conclusions

This functional analysis study performed for the formulated transfer problem revealed the existence of piecewise constant controls. Their calculation indicates values $M = 2.25 \times 10^{-4}$ g, which means that for fuel durations $T \geq 10^5$ s consistent with the hypothesis of small thrust, the amount of the acceleration due to the thrust is situated between the admissible limits of 10^{-3}–10^{-6} g. The obtained commutation time of the command is $t_1 = 2.13 \times 10^5$ s.

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Bang-Bang Control of Flexible Spacecraft Slewing Maneuvers: Guaranteed Terminal Pointing Accuracy

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I. Introduction

FUTURE spacecraft may be quite flexible compared with their predecessors. Many proposed applications require slewing these vehicles between two quiescent attitudes. Bang-bang control of rest-to-rest slewing maneuvers has been the focus of a number of recent investigations.[1-7] In these works the time- and fuel-optimal[1] and the time-optimal[3-7] problems are considered. Fixed-time maneuvers minimizing a measure of spillover energy have also been considered.[2] In some cases it is possible to obtain closed-form expressions for post-time-optimal control spillover measures (spillover energy and maximum postmaneuver pointing error, for example).[5] The purpose of this Note is to extend these results to any bang-bang, rest-to-rest slewing maneuver for a class of flexible spacecraft. The main contribution of this work is that the bounds obtained here allow us to determine, a priori (i.e., before any sequence of switching times is computed), the number of flexible modes to be controlled actively during a bang-bang maneuver in order to guarantee a prespecified pointing accuracy for an infinite dimensional evaluation model.

II. Equations of Motion

We will consider an unconstrained, cylindrical rigid central body to which $N (N \geq 2)$ identical flexible appendages are rigidly and symmetrically attached. The spacecraft is to be controlled by a single torque actuator located at the rigid central body. The appendage displacements and slopes are assumed small relative to the undeformed appendage; appendages are assumed inextensible, and only planar motions are considered. The appendage deformations relative to their undeformed shapes are assumed identical and antisymmetric; no structural damping is assumed. Lastly, it is assumed that the rigid central body rotation rate remains small at all times. Henceforth, we shall use uppercase boldface type to denote matrices and lowercase boldface type to indicate vectors. The following coupled ordinary differential equations are obtained after discretization (using the assumed mode method, for example) and truncation[4,5]:

$$J_0\ddot{\theta} + \boldsymbol{m}^T\ddot{\boldsymbol{q}} = T, \qquad \boldsymbol{m},\boldsymbol{q} \in \mathbb{R}^M \tag{1}$$

$$N\ddot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} + \boldsymbol{m}\ddot{\theta} = 0, \qquad \boldsymbol{m},\boldsymbol{q} \in \mathbb{R}^M \tag{2}$$

where $\theta(t)$ is the rigid-body angular position, $\boldsymbol{q}(t)$ is the appendage generalized coordinate vector, $T(t)$ is the rigid-body control torque, J_0 is the total undeformed rotational inertia of the vehicle, $\boldsymbol{K}$ is a diagonal matrix whose ith diagonal entry is $N\tilde{\omega}_i^2$ ($\tilde{\omega}_i$ is the frequency of the ith bending mode), and $\boldsymbol{m}$ is defined according to

$$\{\boldsymbol{m}\}_i \triangleq N\int_0^L \rho(x)(R + x)\phi_i(x)\,\mathrm{d}x, \qquad i = 1,2,\ldots, M \tag{3}$$

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where $\{\cdot\}_i$ denotes the ith component of the vector argument, $\phi_i(x)$ is the ith orthonormal, assumed mode shape function, $\rho(x)$ is the appendage material density, L is the appendage length, R is the radius of the rigid hub, and M is an arbitrary integer, indicating the order of modal truncation. With the aid of Eq. (1), Eq. (2) can be written as

$$[NI - (1/J_0)mm^T]\ddot{q} + Kq = -(T/J_0)m \tag{4}$$

Let $q = U\eta$, when η is a modal coordinate vector, and U is a normalized modal matrix that satisfies $U^T[NI-(1/J_0)mm^T]U = I$, $U^TKU = \Omega^2 \triangleq \mathrm{diag}(\omega_i^2;\ i = 1,2,\ldots, M)$, with $0<\omega_j<\omega_{j+1}$; $j = 1,2,\ldots, M-1$. The modal coordinates are, therefore, governed by

$$\ddot{\eta} + \Omega^2\eta = -(T/J_0)U^Tm \tag{5}$$

Defining the state variables as

$$x_1 \triangleq \theta + (1/J_0)m^TU\eta \tag{6}$$

$$x_2 \triangleq \dot{\theta} + (1/J_0)m^TU\dot{\eta} \tag{7}$$

$$x_3^i \triangleq \eta_i = \{\eta\}_i, \qquad i = 1,2,\ldots, M \tag{8}$$

$$x_4^i \triangleq \dot{\eta}_i/\omega_i, \qquad i = 1,2,\ldots, M \tag{9}$$

the control variable as

$$u_0 \triangleq T/J_0 \tag{10}$$

and the parameter

$$\beta_0^i \triangleq -\{\Omega^{-1}U^Tm\}_i, \qquad i = 1,2,\ldots, M \tag{11}$$

Equations (1) and (5) can be written as

$$\dot{x}_1 = x_2 \tag{12}$$

$$\dot{x}_2 = u_0 \tag{13}$$

$$\dot{x}_3^i = \omega_i x_4^i, \qquad i = 1,2,\ldots, M \tag{14}$$

$$\dot{x}_4^i = -\omega_i x_3^i + \beta_0^i u_0, \qquad i = 1,2,\ldots, M \tag{15}$$

We now require that only n of the M flexible modes be actively suppressed at the final time; the control model, therefore, is of the same form as Eqs. (12-15), except that the state is $2n+2$ dimensional ($n \le M$). Defining the state vector as $x(t) \triangleq \{x_1, x_2, x_3^1, x_4^1,\ldots, x_3^n, x_4^n\}^T$, the boundary conditions can be expressed as

$$x(0) = \{0,0,\ 0,0,\ldots,\ 0,0\}^T, \qquad x(0)\in \mathbb{R}^{2n+2} \tag{16}$$

$$x(t_f) = \{\theta_f,0,\ 0,0,\ldots,\ 0,0\}^T, \qquad x(t_f)\in \mathbb{R}^{2n+2} \tag{17}$$

where θ_f, the desired slew angle of the rigid central body, is given, and t_f is the maneuver duration.

III. Characterization of the Slewing Control

We assume the following bang-bang control history:

$$u_0(t) = u_0(t)^* \triangleq \begin{cases} U_0, & 0 \le t < t_1 \\ (-1)^j U_0, & t_j \le t < t_{j+1} \quad j = 1,2,\ldots, k-1 \\ (-1)^k U_0, & t_k \le t < t_f \end{cases} \tag{18}$$

where $U_0>0$ ($U_0 \triangleq T_{\max}/J_0$), and k is the number of control switchings. In order to satisfy the boundary conditions, the following equations must be satisfied by the switching times and the final time:

$$(t_f)^2 - 2(t_k)^2 + 2(t_{k-1})^2 - \ldots + 2(-1)^k(t_1)^2$$
$$= 2(-1)^{k+1}\theta_f/U_0 \tag{19}$$

$$(t_f) - 2(t_k) + 2(t_{k-1}) - \ldots + 2(-1)^k(t_1) = 0 \tag{20}$$

$$\cos\{\omega_i t_f\} - 2\cos\{\omega_i t_k\} + \ldots + 2(-1)^k\cos\{\omega_i t_1\} - (-1)^k$$
$$= 0, \qquad i = 1,2,\ldots, n \tag{21}$$

$$\sin\{\omega_i t_f\} - 2\sin\{\omega_i t_k\} + \ldots + 2(-1)^k\sin\{\omega_i t_1\}$$
$$= 0, \qquad i = 1,2,\ldots, n \tag{22}$$

Notice that solving Eqs. (19-22) for switching times and the final time requires knowledge of k, and n, the number of flexible modes to be actively controlled. The basic contribution of this Note is to show how to choose n based on either t_f or k, so that any solution of Eqs. (19-22) yields a bang-bang control such that a prespecified pointing accuracy requirement is satisfied when the control is applied to the infinite-dimensional evaluation model.

IV. Control Spillover: Pointing Error of the Rigid Body

The control characterized in the previous section is now applied to an evaluation model (12-15) with M undamped flexible modes ($M \ge n$) in addition to the rigid-body mode ($M-n$ of these modes are, therefore, not actively controlled). In general, these $M-n$ modes would have nonzero displacements and velocities when the controls are turned off at the end of the maneuver. These effects, therefore, result in free vibration of the system after the maneuver has been completed. We assume that the control is given by

$$\left.\begin{aligned} u_0(t) &= u_0(t)^*, \qquad t\in[0,t_f] \\ &= 0 \quad , \qquad t\notin[0,t_f] \end{aligned}\right\} \tag{23}$$

By integrating Eqs. (14) and (15), we obtain the following expressions for x_3^i, $i = n+1,\ n+2,\ldots, M$, the modal displacements of the uncontrolled modes:

$$x_3^i(t) = \beta_0^i \int_0^{t_f} u_0(\tau)^* \sin\{\omega_i(t-\tau)\}\,d\tau, \qquad t \ge t_f$$
$$i = n+1,\ n+2,\ldots, M \tag{24}$$
$$= -(-1)^k\beta_0^i U_0[\cos\{\omega_i(t-t_f)\} - 2\cos\{\omega_i(t-t_k)\}$$
$$+ \ldots + 2(-1)^k\cos\{\omega_i(t-t_1)\}$$
$$-(-1)^k\cos\{\omega_i t\}]/\omega_i, \qquad t \ge t_f$$
$$i = n+1,\ n+2,\ldots, M$$

We characterize the control spillover via a pointing error of the rigid central body. This is defined as $\theta_e(t) \triangleq \theta(t) - \theta_f$, $t \ge 0$. Inserting Eq. (11) in Eq. (6) and using Eq. (8) we obtain

$$\theta_e(t) \triangleq \theta(t) - \theta_f = x_1(t) + (1/J_0)\sum_{i=1}^{M}(\omega_i\beta_0^i x_3^i) - \theta_f, \quad t \ge 0$$

for $t \ge t_f$

$$\theta_e(t) = (1/J_0)\sum_{i=n+1}^{M}(\omega_i\beta_0^i x_3^i), \qquad t \ge t_f \tag{25}$$
$$= -(-1)^k(U_0/J_0)\sum_{i=n+1}^{M}[[(\beta_0^i)^2[\cos\{\omega_i(t-t_f)\}$$
$$-2\cos\{\omega_i(t-t_k)\} + \ldots + 2(-1)^k\cos\{\omega_i(t-t_1)\}$$
$$-(-1)^k\cos\{\omega_i t\}]]], \qquad t \ge t_f \tag{26}$$

Using Eq. (24) we obtain the following distinct upper bounds on $|x_3^i(t)|$, $t \geq t_f$, $i = n+1,\ldots, M$:

$$|x_3^i(t)| \leq \sqrt{2} U_0 t_f |\beta_0^i|, \qquad t \geq t_f \qquad i = n+1, n+2,\ldots, M \quad (27)$$

$$|x_3^i(t)| \leq (2\sqrt{2}/\pi)\{1 + \pi/(\omega_i t_f)\} U_0 t_f |\beta_0^i|, \qquad t \geq t_f \qquad i = n+1, n+2,\ldots, M \quad (28)$$

Using Eqs. (27) and (28) in (25), the following upper bounds on the postmaneuver pointing error are obtained:

$$|\theta_e(t)| \leq \sqrt{2}(U_0 t_f/J_0) \sum_{i=n+1}^{M} \{\omega_i (\beta_0^i)^2\}, \qquad t \geq t_f \quad (29)$$

$$|\theta_e(t)| \leq (2\sqrt{2}/\pi)(U_0 t_f/J_0) \sum_{i=n+1}^{M} \times \{(\omega_i + \pi/t_f)(\beta_0^i)^2\}, \qquad t \geq t_f \quad (30)$$

Using Eq. (26) yet another upper bound is obtained:

$$|\theta_e(t)| \leq (2k+2)(U_0/J_0) \sum_{i=n+1}^{M} (\beta_0^i)^2, \qquad t \geq t_f \quad (31)$$

These upper bounds can be expressed in closed forms for an infinite dimensional evaluation model where $M = \infty$. The upper bounds obtained above become, respectively,

$$|\theta_e(t)| \leq \sqrt{2}(U_0 t_f/J_0) \sum_{i=n+1}^{\infty} \{\omega_i (\beta_0^i)^2\}, \qquad t \geq t_f \quad (32)$$

$$|\theta_e(t)| \leq (2\sqrt{2}/\pi)(U_0 t_f/J_0) \sum_{i=n+1}^{\infty} \times \{(\omega_i + \pi/t_f)(\beta_0^i)^2\}, \qquad t \geq t_f \quad (33)$$

$$|\theta_e(t)| \leq (2k+2)(U_0/J_0) \sum_{i=n+1}^{\infty} (\beta_0^i)^2, \qquad t \geq t_f \quad (34)$$

Modal identities, the proofs of which rely on the modal relations of Hughes,[8] and that are proven elsewhere,[5] give us the following closed-form expressions:

$$\mathcal{S}_{1\infty} \triangleq \sum_{i=1}^{\infty} (\beta_0^i)^2 = N \int_0^L \left[\rho(y)(R+y) \times \left\{ \int_0^L \rho(x)(R+x)\mathcal{F}(x,y)\, dx \right\} \right] dy \quad (35)$$

$$\mathcal{S}_{2\infty} \triangleq \sum_{i=1}^{\infty} (\omega_i \beta_0^i)^2 = (1 + J_E/J_R) J_E, \qquad J_R \neq 0 \quad (36)$$

$$\mathcal{S}_{3\infty} \triangleq \sum_{i=1}^{\infty} \{\omega_i (\beta_0^i)^2\} \leq [\mathcal{S}_{1\infty}]^{1/2} [\mathcal{S}_{2\infty}]^{1/2} \quad (37)$$

where J_E is the total rotational inertia of the undeformed flexible appendages about the slew axis, J_R is the rotational inertia of the rigid central body about the slew axis, and $\mathcal{F}(x,y)$ is the flexibility kernel[8] associated with the flexible appendage. Using Eqs. (35–37) in Eqs. (32–34), we obtain the following three distinct bounds on the pointing error of the rigid central body:

$$|\theta_e(t)| \leq \sqrt{2}(U_0 t_f/J_0) \left[\mathcal{S}_{3\infty} - \sum_{i=1}^{n} \{\omega_i (\beta_0^i)^2\} \right], \qquad t \geq t_f \quad (38)$$

$$|\theta_e(t)| \leq (2\sqrt{2}/\pi)(U_0 t_f/J_0) \left[\mathcal{S}_{3\infty} - \sum_{i=1}^{n} \{\omega_i (\beta_0^i)^2\} + (\pi/t_f) \times \left\{ \mathcal{S}_{1\infty} - \sum_{i=1}^{n} (\beta_0^i)^2 \right\} \right], \qquad t \geq t_f \quad (39)$$

$$|\theta_e(t)| \leq (2k+2)(U_0/J_0) \left[\mathcal{S}_{1\infty} - \sum_{i=1}^{n} (\beta_0^i)^2 \right], \qquad t \geq t_f \quad (40)$$

Now suppose that $\theta_E \triangleq \max|\theta_e(t)|$, $t \geq t_f$, is specified. Equations (38–40) can be manipulated to yield

$$\sum_{i=1}^{n} \{\omega_i (\beta_0^i)^2\} \leq \mathcal{S}_{3\infty} - [J_0/(\sqrt{2} t_f U_0)]\, \theta_E \quad (41)$$

$$\sum_{i=1}^{n} \{(\omega_i t_f/\pi + 1)(\beta_0^i)^2\} \leq \{(t_f/\pi)\mathcal{S}_{3\infty} + \mathcal{S}_{1\infty}\} - [J_0/(\sqrt{2} t_f U_0)]\, \theta_E \quad (42)$$

$$\sum_{i=1}^{n} (\beta_0^i)^2 \leq \mathcal{S}_{1\infty} - [J_0/\{(2k+2)U_0\}]\, \theta_E \quad (43)$$

Note that Eqs (41) and (42) require knowledge of t_f, the maneuver time, and Eq. (43) requires knowledge of k, the number of control switchings. Once either of these quantities has been specified, these inequalities allow us to determine n, the number of flexible modes to be retained in the control model, such that for a specified number of control switchings or for a specified maneuver time, $|\theta_e(t)| \leq \theta_E$, $t \geq t_f$, is assured. If t_f is fixed then, in order for a solution of Eqs. (19–22) to exist, it is necessary that t_f be chosen greater than or equal to the final time for a time-optimal maneuver.[4,5] When t_f is specified, different values of n are obtained using each of Eq. (41) and (42). The smaller value is chosen to determine the size of the control model.

V. Applications

We now consider some rest-to-rest slewing examples to demonstrate the applicability of the results presented in this Note. Normalized cantilever mode shapes are chosen as the assumed mode shapes. The spacecraft dimension, material, and maneuver specifications are listed in Table 1. The results of the modal analysis for a finite-dimensional ($M = 10$) evaluation model appear in Table 2. It is, however, possible to obtain upper bounds on the postmaneuver pointing error of the rigid central body for an infinite-dimensional evaluation model. For this example $\mathcal{S}_{1\infty} = 192.493$, $\mathcal{S}_{2\infty} = 8244.444$, and $\mathcal{S}_{3\infty} = 1259.763$.

For specified values of θ_E and k, we can obtain the required number of flexible modes to be actively controlled using Eq. (43). Having determined n, the $2n+2$ equations (19–22) can be solved for the control switching times. A solution to these equations can be obtained using a homotopy approach.[4,5,9] In Table 3 we present, columnwise, the specified number of switchings k, the allowable maximum postmaneuver pointing error θ_E, the predicted number of flexible modes to be actively controlled in order to meet the maneuver requirement n, the actual maximum postmaneuver pointing error, maximum $|\theta_e(t)|$, $t \geq t_f$, and the actual number of flexible modes that need to be actively controlled in order to meet the postmaneuver performance objective.

Table 1 Spacecraft dimensions, appendage, and maneuver specifications

Radius of the rigid central body	R	1.00 m
Mass of the rigid central body		50.00 Kg × m²
Number of appendages	N	2
Appendage length	L	5.00 m
Appendage flexural rigidity	EI	328.30 N × m²
Appendage material density	ρ	3.08 Kg/m
Total slewing angle	θ_f	45.00 deg
Maximum available torque	T_{max}	80.00 N × m

Table 2 Modal quantities

ω_i:	3.861	12.013	27.745	51.839	84.241	124.869	173.692	230.692	295.863	369.206
β_0^i:	13.152	4.214	1.221	0.457	0.212	0.114	0.068	0.044	0.030	0.021

Table 3 Predicted and actual control model sizes

	k	θ_E, deg	n^a	$\mathrm{Max}\,|\theta_e(t)|$, $t \geq t_f$, deg^b	n^b
Case 1	3	4.0	1	1.809	1
Case 2	5	2.0	2	0.157	1
Case 3	7	0.2	3	0.040	2

[a]Predicted. [b]Actual.

Table 4 Control switching times

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_f
Case 1	1.844179	2.197564	2.550949	—	—	—	—	4.395128
Case 2	1.821167	2.087151	2.200325	2.313499	2.579483	—	—	4.400650
Case 3	1.819569	2.068951	2.141553	2.200462	2.259371	2.331973	2.581355	4.400924

Note that of the three examples presented here, the predicted and actual sizes of the control model are in agreement for case 1, but for cases 2 and 3 it is possible to meet the stated accuracy requirement by controlling fewer flexible modes than are predicted by Eq. (43). Lastly, in Table 4, we list control switching times and final time, which are solutions of Eqs. (19–22).

VI. Conclusions

The results presented here offer a systematic procedure for determining the size of the control model in order to meet a prespecified postmaneuver pointing accuracy requirement. The evaluation of the size of the control model requires a priori knowledge of either the maneuver duration or the number of control switchings. The predicted size of the control model is not always minimal, i.e., it may be possible to meet the accuracy requirement by controlling fewer modes than are predicted (in our examples, for the stated performance objective, the predicted size is clearly not minimal for cases 2 and 3). The bounds presented here allow us to determine the control model size in order to meet a particular control spillover requirement, namely, the postmaneuver pointing error due to an infinite number of uncontrolled modes. Bounds, similar to the ones presented here, have also been established that allow us to determine the size of the control model such that the residual postmaneuver energy is guaranteed to be less than a prespecified value. Utility of these latter bounds is doubtful since our experience[4] has shown that increasing the order of the control model does not always result in a lower postmaneuver energy.

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